Mixed Linear Complementarity Problem Problems

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1 Introduction

This is a little note showing a problem with solving linear programs (LPs) with *no non-negativity constraints* by converting them to mixed linear complementarity problems (MLCPs) and running the MLCP through Lemke's Algorithm.

2 LP → MLCP Conversion

Here's the LP we're working with:

$$\begin{array}{c} \min\ c^T x \\ Ax \geq b \end{array} \quad \text{where} \quad A = \begin{pmatrix} 1 & 5 \\ 5 & -1 \\ -1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -15 \\ -11 \\ 4 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 10 \end{pmatrix} \tag{1}$$

Note that there is no $x \geq 0$ constraint, and in fact, the solution has both components of x negative.

The solution to this LP[1] is

$$x = \begin{pmatrix} -5/4 \\ -11/4 \end{pmatrix}.$$

To convert this LP into a MLCP, we use the KKT optimality conditions

$$\begin{aligned} u = c - A^T y &= 0, \\ v = Ax - b &\geq 0, \\ (v, y) &\geq 0, \\ \text{and}\ \ v^T y &= 0. \end{aligned}$$

They give us the MLCP

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ -b \end{pmatrix}, \ \ (v, y) \geq 0,\ \ u = 0,\ \text{and } x \text{ free.} \tag{2}$$

[1] Incidentally, this LP was generated by moving the nonnegative LP with $b = (3\ \ 1\ \ -2)^T$ (which is the original LP I sent mail about on 5/24/2001) into the third quadrant by translating x by $(-3\ \ -3)^T$.

3 Lemke's Algorithm

The inital tableau for Lemke's Algorithm for the MLCP (2) is

	1	z_0	x_1	x_2	y_1	y_2	y_3
u_1	1	$-\frac{1}{4}$	0	0	−1	−5	1
u_2	10	$-\frac{5}{2}$	0	0	−5	1	1
v_1	15	1	1	5	0	0	0
v_2	11	1	5	−1	0	0	0
v_3	−4	1	−1	−1	0	0	0

The components of covering vector for the artificial variable (z_0) that correspond to the nonnegativity-constrained basic variables ($v_1, v_2,$ and v_3) are 1, as usual. In this tableau, z_0 will be driven to 4 to create an initial feasible solution by driving v_3 to 0. I've chosen the u components of the covering vector to bring u_1 and u_2 to 0 during this drive, as they should be for a feasible solution. *I just made this up, and I have no idea if it's the right thing to do, so this may part of the problem I outline below.*

Regardless, our first pivot is $\langle v_3, z_0 \rangle$.

	1	v_3	x_1	x_2	y_1	y_2	y_3
u_1	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	−1	−5	1
u_2	0	$-\frac{5}{2}$	$-\frac{5}{2}$	$-\frac{5}{2}$	−5	1	1
v_1	19	1	2	6	0	0	0
v_2	15	1	6	0	0	0	0
z_0	4	1	1	1	0	0	0

(3)

The complement of v_3 is y_3, and we're left with a quandry. Driving y_3 will not increase any of the nonnegative variables because their tableau entries are 0. Driving y_3 *will* cause the u variables to become non-0, however, so we need to pivot one of them into the nonbasic set. But which one? Note that we've numbered this tableau (3) and we'll refer to it later.

Let's pick u_2, for reasons that will become obvious. So, we pivot $\langle u_2, y_3 \rangle$.

	1	v_3	x_1	x_2	y_1	y_2	u_2
u_1	0	$\frac{9}{4}$	$\frac{9}{4}$	$\frac{9}{4}$	4	-6	1
y_3	0	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	5	-1	1
v_1	19	1	2	6	0	0	0
v_2	15	1	6	0	0	0	0
z_0	4	1	1	1	0	0	0

The next complement is x_2. The only option for pivoting is u_1, since driving x_2 will make it non-0. So, we pivot $\langle u_1, x_2 \rangle$.

	1	v_3	x_1	u_1	y_1	y_2	u_2
x_2	0	-1	-1	$\frac{4}{9}$	$-\frac{16}{9}$	$\frac{8}{3}$	$-\frac{4}{9}$
y_3	0	0	0	$\frac{10}{9}$	$\frac{5}{9}$	$\frac{17}{3}$	$-\frac{1}{9}$
v_1	19	-5	-4	$\frac{8}{3}$	$-\frac{32}{3}$	16	$-\frac{8}{3}$
v_2	15	1	6	0	0	0	0
z_0	4	0	0	$\frac{4}{9}$	$-\frac{16}{9}$	$\frac{8}{3}$	$-\frac{4}{9}$

x_1 is now our driving variable. The blocking variable is v_1 (x_2 is free so it can't block us), so we pivot $\langle v_1, x_1 \rangle$.

	1	v_3	v_1	u_1	y_1	y_2	u_2
x_2	$-\frac{19}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{2}{9}$	$\frac{8}{9}$	$-\frac{4}{3}$	$\frac{2}{9}$
y_3	0	0	0	$\frac{10}{9}$	$\frac{5}{9}$	$\frac{17}{3}$	$-\frac{1}{9}$
x_1	$\frac{19}{4}$	$-\frac{5}{4}$	$-\frac{1}{4}$	$\frac{2}{3}$	$-\frac{8}{3}$	4	$-\frac{2}{3}$
v_2	$\frac{87}{2}$	$-\frac{13}{2}$	$-\frac{3}{2}$	4	-16	24	-4
z_0	4	0	0	$\frac{4}{9}$	$-\frac{16}{9}$	$\frac{8}{3}$	$-\frac{4}{9}$

Now we're driving y_1, and we do a minimum ratio test between z_0 and v_2, and the winner is z_0 ($\min\{\frac{4}{16/9}, \frac{87/2}{16}\} = \frac{4}{16/9} = 9/4$). So, we pivot $\langle z_0, y_1 \rangle$.

	1	v_3	v_1	u_1	z_0	y_2	u_2
x_2	$-\frac{11}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{2}$	0	0
y_3	$\frac{5}{4}$	0	0	$\frac{5}{4}$	$-\frac{5}{16}$	$\frac{13}{2}$	$-\frac{1}{4}$
x_1	$-\frac{5}{4}$	$-\frac{5}{4}$	$-\frac{1}{4}$	0	$\frac{3}{2}$	0	0
v_2	$\frac{15}{2}$	$-\frac{13}{2}$	$-\frac{3}{2}$	0	9	0	0
y_1	$\frac{9}{4}$	0	0	$\frac{1}{4}$	$-\frac{9}{16}$	$\frac{3}{2}$	$-\frac{1}{4}$

Since we just pivoted z_0 into the nonbasic set, we're done. We pick out the solution $x_1 = -5/4$, $x_2 = -11/4$ as expected (or hoped). It worked!

But wait!

4 Choosing a Different Path

Back in tableau (3), we made the arbitrary choice to pivot $\langle u_2, y_3 \rangle$ instead of $\langle u_1, y_3 \rangle$. As far as I can tell from looking at the tableau or the history up to that point, there's no reason to pick one or the other. Here's the tableau again for convenience:

	1	v_3	x_1	x_2	y_1	y_2	y_3
u_1	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	-1	-5	1
u_2	0	$-\frac{5}{2}$	$-\frac{5}{2}$	$-\frac{5}{2}$	-5	1	1
v_1	19	1	2	6	0	0	0
v_2	15	1	6	0	0	0	0
z_0	4	1	1	1	0	0	0

(3)

Let's look at what happens when we pivot $\langle u_1, y_3 \rangle$ instead.

	1	v_3	x_1	x_2	y_1	y_2	u_1
y_3	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	1	5	1
u_2	0	$-\frac{9}{4}$	$-\frac{9}{4}$	$-\frac{9}{4}$	-4	6	1
v_1	19	1	2	6	0	0	0
v_2	15	1	6	0	0	0	0
z_0	4	1	1	1	0	0	0

Now the driving complement of u_1 is x_1, so we must pivot $\langle u_2, x_1 \rangle$ to keep u_2 from becoming non-0.

	1	v_3	u_2	x_2	y_1	y_2	u_1
y_3	0	0	$-\frac{1}{9}$	0	$\frac{5}{9}$	$\frac{17}{3}$	$\frac{10}{9}$
x_1	0	-1	$-\frac{4}{9}$	-1	$-\frac{16}{9}$	$\frac{8}{3}$	$\frac{4}{9}$
v_1	19	-1	$-\frac{8}{9}$	4	$-\frac{32}{9}$	$\frac{16}{3}$	$\frac{8}{9}$
v_2	15	-5	$-\frac{8}{3}$	-6	$-\frac{32}{3}$	16	$\frac{8}{3}$
z_0	4	0	$-\frac{4}{9}$	0	$-\frac{16}{9}$	$\frac{8}{3}$	$\frac{4}{9}$

The next complement is x_2, and v_2 is the blocking variable. So, we pivot $\langle v_2, x_2 \rangle$.

	1	v_3	u_2	v_2	y_1	y_2	u_1
y_3	0	0	$-\frac{1}{9}$	0	$\frac{5}{9}$	$\frac{17}{3}$	$\frac{10}{9}$
x_1	$-\frac{5}{2}$	$-\frac{1}{6}$	0	$\frac{1}{6}$	0	0	0
v_1	29	$-\frac{13}{3}$	$-\frac{8}{3}$	$-\frac{2}{3}$	$-\frac{32}{3}$	16	$\frac{8}{3}$
x_2	$\frac{5}{2}$	$-\frac{5}{6}$	$-\frac{4}{9}$	$-\frac{1}{6}$	$-\frac{16}{9}$	$\frac{8}{3}$	$\frac{4}{9}$
z_0	4	0	$-\frac{4}{9}$	0	$-\frac{16}{9}$	$\frac{8}{3}$	$\frac{4}{9}$

Finally, our driving variable is now y_2, the complement of v_2. However, y_2 is unblocked by all basic variables, so we have a secondary ray termination!

5 Questions

1. Was the inital assumption to use z_0 to drive u_1 and u_2 to 0 the correct one?
2. If so, what went wrong here? Was there a good reason to choose the u_2 pivot in (3), which led to the correct answer, over the u_1 pivot, which led to a secondary ray termination?
3. Is this even remotely the way to solve MLCPs? I haven't been able to find a single paper or reference about the actual implementation of a modified Lemke solver for MLCPs, even though everyone says it's trivial.
4. Did I make some stupid mistake and I'm missing something obvious?